

# Dirac-Hestenes spinors and Weierstrass representation for surfaces in 4D complex space

Vadim V. Varlamov

## Abstract

Representations of Dirac-Hestenes and Dirac spinor fields via coordinates of surfaces conformally immersed into 4-dimensional complex space are proposed. A relation between time evolution of spinor fields and integrable deformations of surfaces is discussed.

## 1 Introduction

For the first time Weierstrass representation for conformal immersion of surface into  $\mathbf{R}^3$  appeared in the result of variational problem on search of minimal surface restricted by the some curve [1]. Generalization of Weierstrass formulae for surfaces with mean curvature  $H \neq 0$  was proposed by Eisenhart in 1909 [2]. At present, the interest in this topic is increase after works of Kenmotsu [3] and Konopelchenko [4]. An important relation between integrable deformations of surfaces conformally immersed into  $\mathbf{R}^3$  and nonlinear differential equations of soliton theory has been established in paper [4]. Integrable deformations of surfaces (surfaces of revolution, surfaces of constant mean curvature and so on), which are defined by hierarchies of equations of nonlinear physics, considered in [5]-[10]. A further generalization of Weierstrass representation onto a case of 4-dimensional real spaces with different signatures has been proposed by Konopelchenko and Landolfi in recent papers [11]-[13].

In present paper we consider Weierstrass representation for conformal immersion of surfaces into 4-dimensional complex space. In section 2 generalized Weierstrass formulae in  $\mathbf{C}^4$  are rewritten in a spinor representation

type form which coincides with the matrix representation of a biquaternion algebra  $\mathbf{C}_2 \cong M_2(\mathbf{C})$  known in physics as a Pauli algebra. It allows using the well-known relation between Dirac-Hestenes and Dirac spinors [14, 15] to establish a relation between coordinates for surfaces immersed into  $\mathbf{C}^4$  and Dirac spinors. An equivalence between conjugated spinors and minimal right ideals is given in section 3. Integrable deformations of surfaces in  $\mathbf{C}^4$  defined by Davey-Stewartson hierarchy and their relations with the time evolution of Dirac field are considered in section 4.

## 2 A spinor type form of Weierstrass representation for surfaces in space $\mathbf{C}^4$

Let  $\mathbf{C}^4$  be a 4-dimensional complex space associated with a Dirac algebra  $\mathbf{C}_4$ . Let us consider a generalized Weierstrass representation for immersion of 2-dimensional surfaces in space  $\mathbf{C}^4$ . We propose that generalized Weierstrass formulae in this case have a form [16]

$$\begin{aligned} X^1 &= \frac{i}{2} \int_{\Gamma} (\psi_1 \psi_2 d\bar{z} - \varphi_1 \varphi_2 dz), \\ X^2 &= \frac{1}{2} \int_{\Gamma} (\psi_1 \psi_2 d\bar{z} + \varphi_1 \varphi_2 dz), \\ X^3 &= \frac{1}{2} \int_{\Gamma} (\psi_1 \varphi_2 d\bar{z} - \varphi_1 \psi_2 dz), \\ X^4 &= \frac{i}{2} \int_{\Gamma} (\psi_1 \varphi_2 d\bar{z} + \varphi_1 \psi_2 dz), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \psi_{\alpha} &= p\varphi_{\alpha}, \\ \varphi_{\alpha} &= q\psi_{\alpha}, \end{aligned} \quad \alpha = 1, 2. \tag{2}$$

Here  $\psi_{\alpha}$ ,  $\varphi_{\alpha}$  and  $p$ ,  $q$  are complex-valued functions,  $\Gamma$  is a contour in complex plane  $\mathbf{C}$ . We will interpret the functions  $X^i(z, \bar{z})$  as the coordinates in  $\mathbf{C}^4$ . It is easy to verify that components of induced metric have a form

$$\begin{aligned} g_{zz} &= \overline{g_{z\bar{z}}} = \sum_{i=1}^4 (X_z^i)^2 = 0, \\ g_{z\bar{z}} &= \sum_{i=1}^4 (X_z^i X_{\bar{z}}^i) = \psi_1 \psi_2 \varphi_1 \varphi_2. \end{aligned}$$

Therefore, the formulae (1), (2) define a conformal immersion of surface into  $\mathbf{C}^4$  with induced metric of the form

$$ds^2 = \psi_1 \psi_2 \varphi_1 \varphi_2 dz d\bar{z}.$$

On the other hand, the formulae (1) may be rewritten as follows

$$\begin{aligned} d(X^1 + iX^2) &= i\psi_1\psi_2 d\bar{z}, \\ d(X^1 - iX^2) &= -i\varphi_1\varphi_2 dz, \\ d(X^4 + iX^3) &= i\psi_1\varphi_2 d\bar{z}, \\ d(X^4 - iX^3) &= i\varphi_1\psi_2 dz \end{aligned}$$

or

$$d(X^4\sigma_0 + X^1\sigma_1 + X^2\sigma_2 + X^3\sigma_3) = i \begin{pmatrix} \varphi_1\psi_2 dz & \psi_1\psi_2 \bar{z} \\ \varphi_1\varphi_2 dz & \psi_1\varphi_2 d\bar{z} \end{pmatrix}, \quad (3)$$

where  $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ ,  $\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$  are the matrix representations of units of a quaternion algebra  $\mathcal{C}_{0,2} = \mathbb{H}$ . Since the coordinates  $X^i$  are complex, then it is easy to see that the left part of expression (3) be a biquaternion  $\mathbf{C}_2 = \mathbf{C} \otimes \mathcal{C}_{0,2}$ . Moreover, there is an identity  $\mathbf{C}_2 = \mathcal{C}_{3,0}$ . Indeed, a general element of algebra  $\mathcal{C}_{3,0}$  defined by the following expression

$$\mathcal{A} = a^0 \mathbf{e}_0 + \sum_{i=1}^3 a^i \mathbf{e}_i + \sum_{i=1}^3 \sum_{j=1}^3 a^{ij} \mathbf{e}_{ij} + a^{123} \mathbf{e}_{123}. \quad (4)$$

It is obvious that a volume element  $\omega = \mathbf{e}_{123}$  commutes with all basis elements of algebra  $\mathcal{C}_{3,0}$ . Therefore, a center of  $\mathcal{C}_{3,0}$  consists of the unit  $\mathbf{e}_0$  and element  $\omega$ . Moreover,  $\omega^2 = -1$ , it is allows to identify the volume element with imaginary unit  $i$ . Further, using the obvious identities

$$\begin{aligned} \omega \mathbf{e}_1 &= \mathbf{e}_1 \omega = \mathbf{e}_{23}, \\ \omega \mathbf{e}_2 &= \mathbf{e}_2 \omega = \mathbf{e}_{31}, \\ \omega \mathbf{e}_3 &= \mathbf{e}_3 \omega = \mathbf{e}_{12} \end{aligned}$$

the element (4) may be rewritten in the form

$$\begin{aligned} \mathcal{A} &= a^0 \mathbf{e}_0 + a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^3 \mathbf{e}_3 + a^{12} \mathbf{e}_{12} + a^{31} \mathbf{e}_{31} + a^{23} \mathbf{e}_{23} + a^{123} \mathbf{e}_{123} = \\ &= (a^0 + \omega a^{123}) \mathbf{e}_0 + (a^1 + \omega a^{23}) \mathbf{e}_1 + (a^2 + \omega a^{31}) \mathbf{e}_2 + (a^3 + \omega a^{12}) \mathbf{e}_3. \end{aligned} \quad (5)$$

Recalling that  $i \equiv \omega = \mathbf{e}_{123}$  and suppose  $\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_1$  we obtain  $\mathcal{O}_{3,0} = \mathbf{C}_2$ , where  $\mathbf{C}_2$  is an algebra of complex quaternions, the general element of which has a form

$$\mathcal{A} = c^0\mathbf{e}_0 + c^1\mathbf{e}_1 + c^2\mathbf{e}_2 + c^3\mathbf{e}_{21},$$

where  $c^i \in \mathbf{C}$ . Using the identity  $\mathcal{O}_{3,0} = \mathbf{C}_2$  and denoting  $X^4 = X^0$  we can rewrite the left part of expression (3) as follows

$$\begin{aligned} & (\operatorname{Re} X^0 + \omega \operatorname{Im} X^0)\sigma_0 + (\operatorname{Re} X^1 + \omega \operatorname{Im} X^1)\sigma_1 + \\ & + (\operatorname{Re} X^2 + \omega \operatorname{Im} X^2)\sigma_2 + (\operatorname{Re} X^3 + \omega \operatorname{Im} X^3)\sigma_3 = \\ & = \operatorname{Re} X^0\sigma_0 + \operatorname{Re} X^1\sigma_1 + \operatorname{Re} X^2\sigma_2 + \operatorname{Re} X^3\sigma_3 + \\ & + \operatorname{Im} X^3\sigma_{12} + \operatorname{Im} X^2\sigma_{31} + \operatorname{Im} X^1\sigma_{23} + \operatorname{Im} X^0\sigma_{123}. \end{aligned} \quad (6)$$

Let us consider a space-time algebra  $\mathcal{O}_{1,3}$ . A general element of  $\mathcal{O}_{1,3}$  has a form

$$\mathcal{A} = a^0 + \sum_{i=0}^3 a^i \Gamma_i + \sum_{i=0}^3 \sum_{j=0}^3 a^{ij} \Gamma_{ij} + \sum_{i=0}^3 \sum_{j=0}^3 \sum_{k=0}^3 a^{ijk} \Gamma_{ijk} + a^{0123} \Gamma_{0123},$$

where

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \Gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \Gamma_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (7)$$

It is easy to see that a set  $\mathcal{O}_{1,3}^+$  of all even elements of space-time algebra (sub-algebra of  $\mathcal{O}_{1,3}$ ) is isomorphic to the biquaternion:  $\mathcal{O}_{1,3}^+ \cong \mathcal{O}_{3,0}$ . Further, Dirac algebra  $\mathbf{C}_4$  be a complexification of space-time algebra:  $\mathbf{C}_4 = \mathbf{C} \otimes \mathcal{O}_{1,3}$ . On the other hand, the volume element  $\omega = \mathbf{e}_{01234} \in \mathcal{O}_{4,1}$  is belong to a center of  $\mathcal{O}_{4,1}$  and  $\omega^2 = -1$ , therefore we have an identity  $\mathbf{C}_4 = \mathcal{O}_{4,1}$ .

Consider now an important notion of a minimal left ideal of Clifford algebra. Let  $\mathcal{O}_{p,q}(V, Q)$  be a Clifford algebra over a real field  $\mathbf{R}$ , where  $V$  is a vector space endowed with nondegenerate quadratic form

$$Q = x_1^2 + \dots + x_p^2 - \dots - x_{p+q}^2.$$

A minimal left (respectively right) ideal is a set of type  $I_{p,q} = \mathcal{C}_{p,q}e_{pq}$  (resp.  $e_{pq}\mathcal{C}_{p,q}$ ), where  $e_{pq}$  is a primitive idempotent, i.e.,  $e_{pq}^2 = e_{pq}$  and  $e_{pq}$  cannot be represented as a sum of two orthogonal idempotents, i.e.,  $e_{pq} \neq f_{pq} + g_{pq}$ , where  $f_{pq}g_{pq} = g_{pq}f_{pq} = 0$ ,  $f_{pq}^2 = f_{pq}$ ,  $g_{pq}^2 = g_{pq}$ .

**Theorem 1** (Lounesto [17]). *A minimal left ideal of  $\mathcal{C}_{p,q}$  is of the type  $I_{p,q} = \mathcal{C}_{p,q}e_{pq}$ , where  $e_{pq} = \frac{1}{2}(1 + \mathbf{e}_{\alpha_1}) \dots \frac{1}{2}(1 + \mathbf{e}_{\alpha_k})$  is a primitive idempotent of  $\mathcal{C}_{p,q}$  and  $\mathbf{e}_{\alpha_1}, \dots, \mathbf{e}_{\alpha_k}$  are commuting elements of the canonical basis of  $\mathcal{C}_{p,q}$  such that  $(\mathbf{e}_{\alpha_i})^2 = 1$ ,  $(i = 1, 2, \dots, k)$  that generate a group of order  $2^k$ ,  $k = q - r_{q-p}$  and  $r_i$  are the Radon-Hurwitz numbers, defined by the recurrence formula  $r_{i+8} = r_i + 4$  and*

$$\begin{array}{c|cccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline r_i & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \end{array}.$$

From adduced above theorem immediately follows that minimal left ideals of space-time algebra  $\mathcal{C}_{1,3}$  and Dirac algebra  $\mathcal{C}_{4,1}$  have respectively a form

$$I_{1,3} = \mathcal{C}_{1,3}e_{13} = \mathcal{C}_{1,3}\frac{1}{2}(1 + \Gamma_0), \quad (8)$$

$$I_{4,1} = \mathcal{C}_{4,1}e_{41} = \mathcal{C}_{4,1}\frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_{12}). \quad (9)$$

Moreover, for the minimal left ideal of Dirac algebra using the isomorphisms  $\mathcal{C}_{4,1} = \mathbf{C}_4 = \mathbf{C} \otimes \mathcal{C}_{1,3} \cong M_2(\mathbf{C}_2)$ ,  $\mathcal{C}_{4,1}^+ \cong \mathcal{C}_{1,3} \cong M_2(\mathbb{H})$  and also an identity  $\mathcal{C}_{1,3}e_{13} = \mathcal{C}_{1,3}^+e_{13}$  [14, 18] we have the following expression [19]

$$\begin{aligned} I_{4,1} = \mathcal{C}_{4,1}e_{41} &= (\mathbf{C} \otimes \mathcal{C}_{1,3})e_{41} \cong \mathcal{C}_{4,1}^+e_{41} \cong \mathcal{C}_{1,3}e_{41} = \\ &= \mathcal{C}_{1,3}e_{13}\frac{1}{2}(1 + i\Gamma_{12}) = \mathcal{C}_{1,3}^+e_{13}\frac{1}{2}(1 + i\Gamma_{12}). \end{aligned} \quad (10)$$

Further, let  $\Phi \in \mathcal{C}_{4,1} \cong M_4(\mathbf{C})$  be a Dirac spinor and  $\phi \in \mathcal{C}_{1,3}^+ \cong \mathcal{C}_{3,0} = \mathbf{C}_2$  be a Dirac-Hestenes spinor. Then from (10) immediately follows a relation between spinors  $\Phi$  and  $\phi$ :

$$\Phi = \phi\frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_{12}). \quad (11)$$

Since  $\phi \in \mathcal{C}_{1,3}^+ \cong \mathcal{C}_{3,0}$ , then the Dirac-Hestenes spinor can be represented by a biquaternion number

$$\begin{aligned} \phi &= a^0 + a^{01}\Gamma_{01} + a^{02}\Gamma_{02} + a^{03}\Gamma_{03} + \\ &\quad + a^{12}\Gamma_{12} + a^{13}\Gamma_{13} + a^{23}\Gamma_{23} + a^{0123}\Gamma_{0123}. \end{aligned} \quad (12)$$

Or in the matrix representation

$$\phi = \begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^* \\ \phi_3 & \phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}, \quad (13)$$

where

$$\begin{aligned} \phi_1 &= a^0 - ia^{12}, \\ \phi_2 &= a^{13} - ia^{23}, \\ \phi_3 &= a^{03} - ia^{0123}, \\ \phi_4 &= a^{01} + ia^{02}. \end{aligned}$$

According to (9) and (11), (13) in the matrix representation elements of minimal left ideal of Dirac algebra have a form

$$\Phi = \begin{pmatrix} \phi_1 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 \\ \phi_3 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

Thus, the elements of this minimal left ideal contain four complex, or eight real parameters, which are just sufficient to define a Dirac spinor.

Let us return to a spinor type form of Weierstrass representation (3) and (6). It is easy to see that by force of  $\mathcal{O}_{3,0} \cong \mathcal{O}_{1,3}^+$  and  $\mathcal{O}_{4,1}^{++} \cong \mathcal{O}_{1,3}^+ \cong \mathcal{O}_{3,0}$  the right part of expression (6) is isomorphic to the following biquaternion

$$\begin{aligned} \phi &= \text{Re } X^0 I + \text{Re } X^1 \Gamma_{01} + \text{Re } X^2 \Gamma_{02} + \text{Re } X^3 \Gamma_{03} + \\ &\quad + \text{Im } X^3 \Gamma_{12} + \text{Im } X^2 \Gamma_{31} + \text{Im } X^1 \Gamma_{23} + \text{Im } X^0 \Gamma_{0123}, \end{aligned} \quad (15)$$

which may be rewritten in the matrix form (13) if suppose

$$\begin{aligned} \phi_1 &= \text{Re } X^0 - i \text{Im } X^3, \\ \phi_2 &= \text{Im } X^2 - i \text{Im } X^1, \\ \phi_3 &= \text{Re } X^3 - i \text{Im } X^0, \\ \phi_4 &= \text{Re } X^1 + i \text{Re } X^2, \end{aligned} \quad (16)$$

So, we establish a relation between Weierstrass coordinates for surfaces conformally immersed into  $\mathbf{C}^4$  and Dirac-Hestenes spinors. Further, in accordance with (11) it is easy to establish a relation with the Dirac spinor

$\Phi \in M_4(\mathbf{C})e_{41}$  treated as minimal left ideal of  $\mathcal{C}_{4,1} = \mathbf{C}_4 \cong M_4(\mathbf{C})$ . Therefore, a Dirac field  $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$  (which as known described an electron in physics) may be expressed by means of relations (16) via the generalized Weierstrass formulae (1). In some sense it is allows to consider the electron as a surface conformally immersed into 4-dimensional complex space  $\mathbf{C}^4$ .

### 3 Charge conjugation and antiautomorphism

$$\mathcal{A} \longrightarrow \widetilde{\mathcal{A}}^\star$$

In Clifford algebra  $\mathcal{C}_{p,q}$  there exist four fundamental automorphisms [20, 21]:

1) An automorphism  $\mathcal{A} \rightarrow \mathcal{A}$ .

This automorphism, obviously, be an identical automorphism of algebra  $\mathcal{C}_{p,q}$ ,  $\mathcal{A}$  is an arbitrary element of  $\mathcal{C}_{p,q}$ .

2) An automorphism  $\mathcal{A} \rightarrow \mathcal{A}^\star$ .

In more details, for arbitrary element  $\mathcal{A} \in \mathcal{C}_{p,q}$  there exist a decomposition

$$\mathcal{A} = \mathcal{A}' + \mathcal{A}'',$$

where  $\mathcal{A}'$  is an element consisting of homogeneous odd elements, and  $\mathcal{A}''$  is an element consisting of homogeneous even elements, respectively. Then the automorphism  $\mathcal{A} \rightarrow \mathcal{A}^\star$  is that element  $\mathcal{A}''$  is not changed, and element  $\mathcal{A}'$  is changed the sign:

$$\mathcal{A}^\star = -\mathcal{A}' + \mathcal{A}''.$$

If  $\mathcal{A}$  is a homogeneous element, then

$$\mathcal{A}^\star = (-1)^k \mathcal{A}, \tag{17}$$

where  $k$  is a degree of element.

3) An antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ .

The antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$  be a reversion of the element  $\mathcal{A}$ , that is the substitution of the each basis element  $\mathbf{e}_{i_1 i_2 \dots i_k} \in \mathcal{A}$  by the element  $\mathbf{e}_{i_k i_{k-1} \dots i_1}$ :

$$\mathbf{e}_{i_k i_{k-1} \dots i_1} = (-1)^{\frac{k(k-1)}{2}} \mathbf{e}_{i_1 i_2 \dots i_k}.$$

Therefore, for any  $\mathcal{A} \in \mathcal{C}_{p,q}$  we have

$$\widetilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}} \mathcal{A}. \tag{18}$$

4) An antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$ .

This antiautomorphism be a composition of the antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$  with the automorphism  $\mathcal{A} \rightarrow \mathcal{A}^*$ . In the case of homogeneous element from formulae (17) and (18) follows

$$\widetilde{\mathcal{A}}^* = (-1)^{\frac{k(k+1)}{2}} \mathcal{A}. \quad (19)$$

It is obvious that  $\widetilde{\widetilde{\mathcal{A}}} = \mathcal{A}$ ,  $(\mathcal{A}^*)^* = \mathcal{A}$ ,  $(\widetilde{\mathcal{A}}^*)^* = \mathcal{A}$ .

The antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  is present for us a most interest, since this antiautomorphism is closely related with a charge conjugation in theory of electron [21]. It is known [21] that in the matrix representation the antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  defined by the following expression

$$\widetilde{\mathcal{A}}^* = (CE^T)A^T(CE^T)^{-1}, \quad (20)$$

where  $E$  is a matrix of volume element of  $\mathcal{C}_{p,q}$ ,  $C$  is a matrix of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ . Let us now find a matrix  $CE^T$  of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  for the space-time algebra  $\mathcal{C}_{1,3}$ . In the spinbasis (7) the matrix  $E$  has a form

$$E = \Gamma_{0123} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

Further, under action of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$  the units of  $\mathcal{C}_{p,q}$  are transfered into themselves:  $\mathbf{e}_i \rightarrow \mathbf{e}_i$  ( $i = 1, \dots, n = p + q$ ). Therefore, in the case of spinbasis (7) we have also:  $\Gamma_i \rightarrow \Gamma_i$ . On the other hand, in the matrix representation for  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$  we have [21]

$$A \longrightarrow CA^TC^{-1}. \quad (21)$$

Transposition of the matrices (7) gives

$$\Gamma_0^T = \Gamma_0, \quad \Gamma_1^T = -\Gamma_1, \quad \Gamma_2^T = \Gamma_2, \quad \Gamma_3^T = -\Gamma_3.$$

Take into account the last relations we obtain from (21)

$$\begin{aligned} \Gamma_0 &\longrightarrow \Gamma_0 = C\Gamma_0C^{-1}, & \Gamma_1 &\longrightarrow \Gamma_1 = -C\Gamma_1C^{-1}, \\ \Gamma_2 &\longrightarrow \Gamma_2 = C\Gamma_2C^{-1}, & \Gamma_3 &\longrightarrow \Gamma_3 = -C\Gamma_3C^{-1}, \end{aligned} \quad (22)$$



or

$$\begin{aligned}\Gamma_0 C &= C \Gamma_0, & \Gamma_1 C &= -C \Gamma_1, \\ \Gamma_2 C &= C \Gamma_2, & \Gamma_3 C &= -C \Gamma_3.\end{aligned}\tag{23}$$

It is easy to see that a matrix  $C = \Gamma_{13}$  is satisfy to conditions (23) and therefore be a matrix of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$  for the spinbasis (6). Hence it immediately follows that for a matrix of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  we have

$$CE^T = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Further, using the matrix representation (20) we find that an action of anti-automorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  on Dirac-Hestenes spinor field is expressed as follows

$$\widetilde{\phi}^* = \begin{pmatrix} \phi_1^* & \phi_2^* & -\phi_3^* & -\phi_4^* \\ -\phi_2 & \phi_1 & -\phi_4 & \phi_3 \\ -\phi_3^* & -\phi_4^* & \phi_1^* & \phi_2^* \\ -\phi_4 & \phi_3 & -\phi_2 & \phi_1 \end{pmatrix}.\tag{24}$$

Whence, in accordance with relations (11) for a charge conjugated Dirac spinor we obtain

$$\begin{aligned}\widetilde{\Phi}^* &= (\widetilde{\phi e_{41}})^* = \widetilde{e_{41}^*} \widetilde{\phi}^* = \\ &\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1^* & \phi_2^* & -\phi_3^* & -\phi_4^* \\ -\phi_2 & \phi_1 & -\phi_4 & \phi_3 \\ -\phi_3^* & -\phi_4^* & \phi_1^* & \phi_2^* \\ -\phi_4 & \phi_3 & -\phi_2 & \phi_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\phi_4 & \phi_3 & -\phi_2 & \phi_1 \end{pmatrix}.\end{aligned}\tag{25}$$

Therefore, *Dirac spinor (14) treated as a minimal left ideal of algebra  $\mathcal{C}_{4,1} = \mathbf{C}_4 \cong M_4(\mathbf{C})$  under action of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  is transfered into a minimal right ideal of type (25).* Therefore, the field  $\widetilde{\Phi}^* = (-\phi_4, \phi_3, -\phi_2, \phi_1)$  is expressed by means of (16) via the coordinates of surface conformally immersed into  $\mathbf{C}^4$ . Thus, we have the following

**Theorem 2.** *Under action of antiautomorphism  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^\star$  of space-time algebra  $\mathcal{C}\ell_{1,3}$  the Dirac spinor*

$$\Phi = \phi e_{41} = \begin{pmatrix} \phi_1 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 \\ \phi_3 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 \end{pmatrix}$$

*which be a minimal left ideal of Dirac algebra  $\mathcal{C}\ell_{4,1} = \mathbf{C}_4 \cong \mathbf{M}_4(\mathbf{C})$ , is transferred into a minimal right ideal (a charge conjugated spinor)*

$$\widetilde{\Phi}^\star = \widetilde{e_{41}}^\star \widetilde{\phi}^\star = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\phi_4 & \phi_3 & -\phi_2 & \phi_1 \end{pmatrix}$$

*of the same algebra. Here  $\phi$  is a Dirac-Hestenes spinor,  $e_{41} = \frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_{12})$  is a primitive idempotent of  $\mathbf{C}_4$ , and*

$$\begin{aligned} \phi_1 &= \operatorname{Re} X^0 - i\operatorname{Im} X^3, \\ \phi_2 &= \operatorname{Im} X^2 - i\operatorname{Im} X^1, \\ \phi_3 &= \operatorname{Re} X^3 - i\operatorname{Im} X^0, \\ \phi_4 &= \operatorname{Re} X^1 + i\operatorname{Re} X^2, \end{aligned}$$

*where*

$$\begin{aligned} X^1 &= \frac{i}{2} \int_{\Gamma} (\psi_1 \psi_2 d\bar{z} - \phi_1 \phi_2 dz), \\ X^2 &= \frac{1}{2} \int_{\Gamma} (\psi_1 \psi_2 d\bar{z} + \phi_1 \phi_2 dz), \\ X^3 &= \frac{1}{2} \int_{\Gamma} (\psi_1 \phi_2 d\bar{z} - \phi_1 \psi_2 dz), \\ X^0 &= \frac{i}{2} \int_{\Gamma} (\psi_1 \phi_2 d\bar{z} + \phi_1 \psi_2 dz) \end{aligned}$$

*are generalized Weierstrass formulae for conformal immersion of surface into  $\mathbf{C}^4$ .*

## 4 Integrable deformations and time evolution of Dirac field

The system (2) is known in soliton theory as a Davey-Stewartson II (DSII) linear problem [22, 23]. Integrable deformations of surface conformally immersed into  $\mathbf{C}^4$  are defined by an infinite hierarchy of nonlinear differential equations associated with the system (2). This hierarchy is appear as compatibility conditions of (2) with systems of the following form

$$\begin{aligned}\psi_{\alpha t_n} &= A_n \psi_\alpha + B_n \varphi_\alpha, \\ \varphi_{\alpha t_n} &= C_n \psi_\alpha + D_n \varphi_\alpha,\end{aligned}\quad \alpha = 1, 2, \quad (26)$$

where  $t_n$  are new deformation variables and  $A_n, B_n, C_n, D_n$  are differential operators of  $n$ -th order. For example, in the case  $n = 3$  there is the system

$$\begin{aligned}p_{t_3} &= p_{zzz} + p_{\bar{z}\bar{z}\bar{z}} + 3p_z \partial_{\bar{z}}^{-1}(pq)_z + 3p_{\bar{z}} \partial_z^{-1}(qp)_{\bar{z}} + 3p \partial_{\bar{z}}^{-1}(qp_z)_z + 3p \partial_z^{-1}(qp_{\bar{z}})_{\bar{z}}, \\ q_{t_3} &= q_{zzz} + q_{\bar{z}\bar{z}\bar{z}} + 3q_z \partial_{\bar{z}}^{-1}(pq)_z + 3q_{\bar{z}} \partial_z^{-1}(pq)_{\bar{z}} + 3q \partial_{\bar{z}}^{-1}(pq_z)_z + 3q \partial_z^{-1}(pq_{\bar{z}})_{\bar{z}}\end{aligned}\quad (27)$$

and

$$\begin{aligned}A_3 &= \partial_{\bar{z}}^3 + 3[\partial_z^{-1}(pq)_{\bar{z}}] \partial_{\bar{z}} + 3[\partial_z^{-1}(qp_{\bar{z}})]_{\bar{z}}, \\ B_3 &= -p \partial_z^2 + p_z \partial_z - p_{zz} - 3p[\partial_{\bar{z}}^{-1}(pq)_z], \\ C_3 &= -q \partial_{\bar{z}}^2 + q_{\bar{z}} \partial_{\bar{z}} - q_{\bar{z}\bar{z}} - 3q[\partial_z^{-1}(pq)_{\bar{z}}], \\ D_3 &= \partial_z^3 + 3[\partial_{\bar{z}}^{-1}(pq)_z] \partial_z + 3[\partial_{\bar{z}}^{-1}(pq)_{\bar{z}}],\end{aligned}\quad (28)$$

It is obvious that deformation of  $\psi_\alpha, \varphi_\alpha$  by means of (26) induces deformations of coordinates  $X^i(z, \bar{z}, t_n)$  of surface in  $\mathbf{C}^4$ . Moreover, according to (16) DSII-deformation generates a deformation (time evolution) of Dirac-Hestenes spinor field and in accordance with (14) a time evolution of Dirac field  $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ . The all chain of deformations may be represented by the following scheme

$$p, q \longrightarrow \psi_\alpha, \varphi_\alpha \longrightarrow X^i \longrightarrow \phi_i \longrightarrow \Phi.$$

**Proposition 1.** *A time evolution of the Dirac fields  $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$  and  $\widetilde{\Phi}^* = (-\phi_4, \phi_3, -\phi_2, \phi_1)$ , where the components  $\phi_i$  are expressed via coordinates of conformally immersed surface into complex space  $\mathbf{C}^4$ , is defined by the DSII-hierarchy.*

In particular case  $p = q$  and  $p = \bar{p}$  the equations (27), reduce to a modified Veselov-Novikov equation [24]

$$p_t = p_{zzz} + p_{\bar{z}\bar{z}\bar{z}} + 3p_z\partial_{\bar{z}}^{-1}(|p|_z^2) + 3p_{\bar{z}}\partial_z^{-1}(|p|_{\bar{z}}^2) + 3p\partial_{\bar{z}}^{-1}(\bar{p}p_z)_z + 3p\partial_z^{-1}(\bar{z}p_{\bar{z}})_{\bar{z}}.$$

Therefore, *integrable deformations of surfaces in  $\mathbf{C}^4$  with  $p = q$ ,  $p = \bar{p}$  are defined by the mVN-hierarchy.*

In other particular case  $q = 1$  and  $p$  is a real-valued function the equations (27), reduce to Veselov-Novikov equation [25, 26]

$$p_t = p_{zzz} + p_{\bar{z}\bar{z}\bar{z}} + 3[p\partial_{\bar{z}}^{-1}(p_z)]_z + 3[p\partial_z^{-1}(p_{\bar{z}})]_{\bar{z}}.$$

So, *integrable deformations of surfaces in  $\mathbf{C}^4$  with  $q = 1$  are generated by the Veselov-Novikov hierarchy.*

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Vadim V. Varlamov  
 Siberia State University of Industry  
 Kirova 42  
 Novokuznetsk 654007  
 Russia  
 E-mail: root@varlamov.kemerovo.su